Brane & Quantization

Reference: Brane & Quantization" by Gukov-Witten (2008)

(I) Quantization

§ Quantum mechanics

$X = T^*M$, $\omega_{can} = \sum dx^j \wedge dp_j$ sympl.

$x, p \in C^\infty(X)$ commuting

Uncertainty principle: $[\hat{x}, \hat{p}] \sim i\hbar$ $\rightsquigarrow$ non-comm. ring.

$$\left([\hat{f}, \hat{g}] = -i\hbar \widehat{\{f, g\}} + O(\hbar^2) \quad \forall f, g \in C^\infty(X) \right.$$

$\{f, g\}$ = Poisson bracket

- deform $(C^\infty(X), \cdot)$ to $(C^\infty(X)[\![\hbar]\!], *_\hbar)$.

  non-comm. ring

  – Deformation Quantization

- Change "functions to (unitary) operators"

$$C^\infty(X) \longrightarrow \mathrm{End}(\mathcal{H}) \quad \text{some Hilbert space}$$
$$f \longmapsto \hat{f}$$

$$*_\hbar \longleftrightarrow \circ \quad \text{composition}$$

– $\mathcal{H}$ Geometric Quantization.

Eg. $X = T^*\mathbb{R}^n$, $\hat{x} = x\cdot$ & $\hat{p} = i\hbar \frac{\partial}{\partial x}$

acting on $\mathbb{C}[x_1, \cdots, x_n] \subset L^2(\mathbb{R}^n) = \mathcal{H}$

§ Geometric Quantization

(X, ω) Need $[\omega] \in H^2(X, \mathbb{Z})$

$\Rightarrow \mathbb{C} \to \mathcal{L} \longrightarrow X$ s.t. $F_{\mathcal{L}} = \omega$

$\mathcal{H}$ should be 'functions' on x, not x, p

Instead of $(x, \cancel{p})$, use $(\underset{x+ip}{z}, \underset{x-ip}{\cancel{\bar{z}}})$, i.e. holomorphic sections

Choose J s.t. (X, J, ω) Kähler.

set $\mathcal{H} = H^0(X, K_X^{\frac{1}{2}} \otimes \mathcal{L})$ (or $H^*(X, K_X^{\frac{1}{2}} \otimes \mathcal{L})$)

§ Represention of Compact Lie groups.

" $G \curvearrowright (X, \omega/\mathbb{Z}) \rightsquigarrow G \curvearrowright H^0(X, \mathcal{L})$ "

Eg. G <u>compact</u> Lie group (say $\pi_1 = 0$)

$G \curvearrowright \mathfrak{g}^* \supset \mathcal{O}_\lambda$ coadj. orbit at $\lambda \in \mathfrak{t}^* \subset \mathfrak{g}^*$

$H^2(\mathcal{O}_\lambda, \mathbb{R}) \simeq \mathfrak{t}^*$

$\exists$ canon. sympl. $[\omega_\lambda] \longleftrightarrow \lambda$

$\exists \mathcal{L}_\lambda \leftsquigarrow \omega_\lambda / \mathbb{Z} \Longleftrightarrow \lambda \in \mathfrak{t}^*_{\mathbb{Z}}$

$\rightsquigarrow G \curvearrowright H^0(\mathcal{O}_\lambda, \mathcal{L}_\lambda)$ is repr. of G

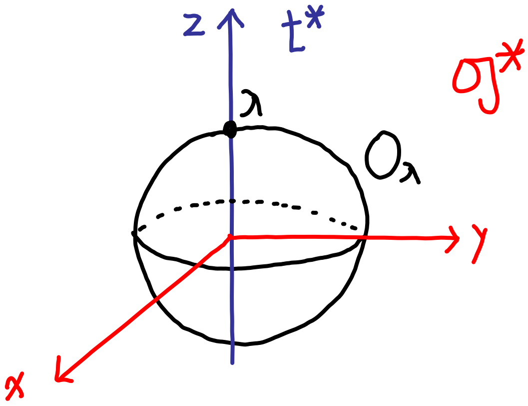
In fact, this gives ALL irred. rep. of G.

Eg $G = SU(2) \curvearrowright \mathfrak{g}^* = \mathbb{R}^3 \supset \mathcal{O}_\lambda \ni \lambda \in \mathbb{R}_z = \mathfrak{t}^*$

$$\mathcal{O}_\lambda = \{x^2+y^2+z^2=\lambda^2\} = S^2 = \mathbb{CP}^1$$

$0 < \lambda \in \mathfrak{t}^*_{\mathbb{Z}} = \mathbb{Z} \rightsquigarrow \mathcal{L}_\lambda = \mathcal{O}(\lambda) \longrightarrow \mathbb{CP}^1$

$\rightsquigarrow SU(2) \curvearrowright H^0(\mathbb{CP}^1, \mathcal{O}(\lambda)) = S^\lambda \mathbb{C}^2$ irred.



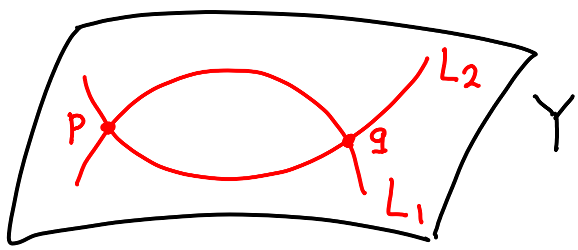
§ Lagr. A-brane $(L, \mathcal{L})$ in (Y, ω_Y) Sympl.

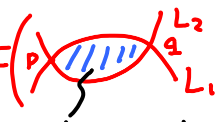
$L \underset{\text{Lagr.}}{\subset} Y$ & $\mathcal{L}$: flat U(1)-bdl./L $\rightsquigarrow$ object in $\mathcal{F}uk(Y)$

$\mathrm{Hom}_{\mathcal{F}(Y)}(L_1, L_2)$ (Say $\mathcal{L}_i$ trivial)

$\equiv HF(L_1, L_2)$ Floer homology

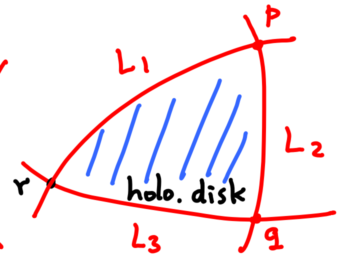
$\triangleq H^\bullet\big(\bigoplus_{p \in L_1 \cap L_2} \mathbb{C}\langle p \rangle, \ \delta\big)$
↑ count #(holo. disk/instanton).



$\delta p = \sum_q \#\Big($  $\Big) \cdot q$

- $\mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_2, L_3) \longrightarrow \mathrm{Hom}(L_1, L_3)$

$$p \otimes q \longmapsto \sum \underbrace{C_r}\cdot r$$

$\#\Big($  $\Big)$ count holo. Δ.

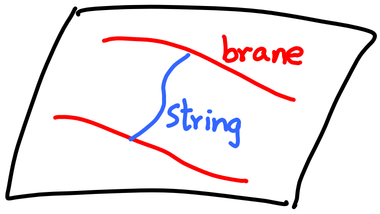
In particular,

1) $\mathrm{Hom}(L, L)$ is algebra,

2) $\mathrm{Hom}(L, L')$ is $\mathrm{Hom}(L,L)$-$\mathrm{Hom}(L',L')$ bi-mod.

Remark : Branes are boundary values of strings

Lagr. condit$^{\underline{n}}$ $\Rightarrow$ preserve enough SUSY



Kapustin : $\exists$ generalization : Coisotropic A-branes.

(much stronger than just coisotropic)

Recall $C^{n+k} \subset (Y^{2n}, \omega)$ coisotropic

$\Leftrightarrow$ each $T_x C \subset T_x Y$ $\exists$ sympl. coord.

s.t. $\underbrace{\underbrace{x_1\ x_2 \cdots x_n \,\vdots\, y_1\ y_2}_{T_x C} \cdots\cdots y_n}_{T_x Y}$

Roughly

$$\begin{array}{ccccc} & & & & x_{k+1}\cdots x_n \\ C & = & C_1 & \times & C_2 \\ \cap & & \| & & \cap \text{ Lagr.} \\ Y & = & Y_1 & \times & Y_2 \\ & & x_1 \cdots x_k & & x_{k+1} \cdots x_n \\ & & y_1 \cdots y_k & & y_{k+1} \cdots y_n \end{array}$$

§ Coisotropic A-brane $(C, \mathcal{L})$ in (Y, ω_Y) Sympl.

Roughly speaking,

$$\begin{array}{ccccc} C & = & C_1 & \times & C_2 \\ \cap & & \| & & \cap^{\text{Lagr.}} \\ Y & = & Y_1 & \times & Y_2 \end{array} \qquad \& \qquad \mathcal{L} = \underset{?}{\mathcal{L}_1} \boxtimes \underbrace{\mathcal{L}_2}_{\text{flat}}$$

So, assume $C = Y$, require $F_{\mathcal{L}} =: \omega_J$

s.t. $\Omega := \omega_J + i \underbrace{\omega_K}_{\omega_Y}$ I-holo sympl. form

(where $I := \omega_K^{-1} \circ \omega_J$)

Namely, $(Y, \Omega = \omega_J + i\omega_K)$ cpx. sympl. mfd & $[\omega_J]/\mathbb{Z}$

$\Rightarrow (Y, \omega_J = F_{\mathcal{L}})$ coiso. A-brane in (Y, ω_K)

called canonical coisotropic brane $\mathcal{B}_{c.c.}$

Witten claim: 1) $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = H^*_{\bar\partial_I}(Y, \mathcal{O}_Y)$

will only use $* = 0$ part
i.e. I-holom. fu. on Y.

w/ alg. str. $= I$-holo. deformation quant. w.r.t. Ω

ASSUME $\omega_J|_M$ non-degenerate.

2) $\mathrm{Hom}(\mathcal{B}_{cc}, \underbrace{M}_{\omega_K\text{-Lagr.}}) =$ Geom. Quant. of (M, ω_J)

Assume Y <u>Hyperkähler</u> $(\omega_I, \underbrace{\omega_J}_{F_{\mathcal{L}}}, \omega_K)$

$\Rightarrow$ $\mathcal{B}_{cc}$, M both (A, B, A) branes

Before $\mathcal{H} := \mathrm{Hom}(\mathcal{B}_{cc}, M)$ via A-model for ω_K

Same $\mathcal{H}'$ via B-model for J ($\because$ {zero energy states in σ-model}).

$$\mathcal{H}' = H^*_J(M, K^{1/2} \otimes \mathcal{L})$$

- $\mathcal{H}$, $\mathcal{H}'$: same as vector spaces,

  w/ different $\mathbb{Z}$-gradings + inner products

Remark: Given any (analy) sympl. $(M, \omega_J = F_{\mathcal{L}})$

$\Rightarrow$ $Y = M_{\mathbb{C}} = T^*M$ is I-cpx. sympl. $\Omega = \omega_J + i\omega_K$

$\cup$ (near $M \subset Y$)

M ω_K-Lagr. in Y

- $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \curvearrowright \mathrm{Hom}(\mathcal{B}_{cc}, M)$

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) & & \mathrm{Hom}(\mathcal{B}_{cc}, M) \\ \cup & & \| \\ \mathcal{O}_Y(Y) = C^\omega(M) & & H^0(M, \mathcal{L}^{\otimes k}) \end{array} \qquad k \in \mathbb{N}$$

(only acts asym. as $k \to \infty$, Toeplitz quantization)

§ Unitarity

$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ Hilbert space ?

$\exists$ natural $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2) \simeq \mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_1)^*$.

(via 2-point function on the disc.)

So we need isom to conjugate A-model.

Mathematically, need

$$\tau : Y \longrightarrow Y, \quad \tau^2 = 1, \quad \tau^* \omega_K = -\omega_K.$$

- $\forall$ τ-inv. compat. metric g ($\rightsquigarrow K$),

$\Rightarrow \tau$ is anti-holomorphic involution w.r.t. K.

$\mathcal{B}_{cc}$: τ-inv. $\Longrightarrow$ ω_J : τ-inv.

(Hence, $\tau^* \Omega = \overline{\Omega}$)

$\Longrightarrow$ $\tau^* I = -I$ $\begin{pmatrix} \text{i.e. anti-holo.} \\ \text{w.r.t. } I \end{pmatrix}$

Lagr $\mathcal{B}'$: τ-inv. $\Longrightarrow$ $\tau(M) = M$

(more general than $M \subset Y^\tau$).

In fact, only really need

$$\tau(M) \underset{\text{Hamil. isotopy}}{\sim} M$$

Hamil. isotopy (asym. =).

§ $G_{\mathbb{C}} = SO(3,\mathbb{C}) \curvearrowright \mathfrak{g}_{\mathbb{C}}^* = \mathbb{C}^3_{x,y,z}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup\!\shortmid$

complex coadj. orbit : $Y := \mathcal{O}_\mu = \{ \underbrace{x^2+y^2+z^2-\frac{\mu^2}{4}}_{f(x,y,z)} = 0 \}$

$$\Omega = \frac{1}{h}\frac{dy \wedge dz}{x} = \frac{2}{h}\mathrm{Res}\left(\frac{dx\wedge dy\wedge dz}{f}\right) :$$ $G_{\mathbb{C}}$-inv. holo. sympl. form

Claim: $\frac{1}{2\pi}\underbrace{\mathrm{Re}\,\Omega}_{F_{\mathcal{L}}} / \mathbb{Z} \iff n := \mathrm{Re}(h^{-1}\mu) \in \mathbb{Z}$

[reason : $H_2(Y,\mathbb{Z}) = \mathbb{Z}\langle S\rangle$ w/ $S = Y \cap \mathbb{R}^3 \approx S^2$
and $\frac{1}{2\pi}\int_S \Omega = h^{-1}\mu$ $\square$

$\Longrightarrow$ $\mathcal{B}_{cc} = (Y, \mathcal{L})$ coiso. A-brane in $(Y, \overbrace{\mathrm{Im}\,\Omega}^{\omega_Y})$.

$$\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = \mathcal{O}_Y(Y)$$

$$\mathrm{Hom}^{\text{classical}}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = \mathcal{O}_Y(Y) = \mathbb{C}[x,y,z]/\langle f\rangle$$

$$\mathrm{Hom}^{\text{quantum}}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = \mathbb{C}\{x,y,z\}/\langle f',\ [x,y] = \alpha(\mu^2,h)z, \ldots\rangle$$

such constraints ($\Leftarrow$ $G_{\mathbb{C}}$-symmetry + holomorphicity

asym. scaling $\Rightarrow$ $\alpha(\mu^2, h) = Ch$

compat. w/ $\{\ \}$ $\Rightarrow$ $[x,y] = hz$ (and Ω)

Similarly, $f' = f - \frac{1}{4}h^2$

Via symmetry $(\mu^2, h) \mapsto (t^2\mu^2, th)$, assume $h=1$.

$$\Rightarrow \quad \mathrm{Hom}^{qu}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) / \langle \underbrace{x^2+y^2+z^2}_{J^2} = \frac{\mu^2-1}{4}\rangle$$

Recall

- $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \triangleq \mathcal{T}(\mathfrak{g}_{\mathbb{C}}) / \langle u\otimes v - v \otimes u - [u,v] \rangle$

  univ. enveloping alg. tensor alg.

- repr. of Lie alg. $\mathfrak{g}_{\mathbb{C}}$ $\Longleftrightarrow$ repr. of assoc. alg. $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$

- $J^2 \in \mathrm{Center}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}}))$ quadratic Casimir operator.

- $\mathfrak{g}_{\mathbb{C}} \curvearrowright V$ irred. $\Longrightarrow$ $J^2 = \text{const}\cdot$ on V

  (eg. $(\mu^2-1)/4$ in our case).

- $\forall$ A-brane $\mathcal{B}'$, $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \curvearrowright \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$

$$\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) / J^2 = (\mu^2-1)/4$$

i.e. $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ is $\mathfrak{g}_{\mathbb{C}}$-repr. w/ $J^2 = (\mu^2-1)/4$

- Hermitian str. on $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$

  require anti-holo. involut^{n} $\tau: Y \longrightarrow Y$ w/ $\tau^*\Omega = \overline{\Omega}$

Eg(1) $\tau(x,y,z) = (\bar{x}, \bar{y}, \bar{z}) \rightsquigarrow SO(3) \leq G_{\mathbb{C}}$

Eg(2) $\tilde{\tau}(x,y,z) = (-\bar{x}, -\bar{y}, \bar{z}) \rightsquigarrow SO(1,2) \leq G_{\mathbb{C}}$, $SO(1,2) = SL(2,\mathbb{R})/\mathbb{Z}_2$

- Y is hyperkähler, can couple w/ B-field (skip).
  (Eguchi-Hansen mfd.)
  ω_I, ω_J, $\omega_K =: \omega_Y$ ← for our A-model.

- Need $\mu^2 \in \mathbb{R}$ $\rightsquigarrow$ τ or $\tilde{\tau}$ preserves Y

  But $\mu = \underbrace{\frac{1}{2\pi}\int_S \omega_J}_{\beta = n \in \mathbb{Z}} + i \underbrace{\frac{1}{2\pi}\int_S \omega_K}_{\gamma}$

  $$\Rightarrow \begin{cases} \gamma = 0 \ \& \ \mu^2 = \beta^2 \geq 0, & \text{or} \\ \beta = 0 \ \& \ \mu^2 = -\gamma^2 \leq 0. \end{cases}$$

---

§ Repr. of $SU(2)$

$\mathcal{B}'$: $M := Y^\tau = Y_{\mathbb{R}} = \{x^2+y^2+z^2 = \frac{\mu^2}{4}\} \cap \mathbb{R}^3$

$M \neq \phi$, pt. $\Rightarrow$ $\mu^2 > 0$ $\Rightarrow$ $\gamma = 0$ & $\beta \neq 0$

- M is $SU(2)$-inv.

  $\Rightarrow SU(2) \curvearrowright \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ ← quantizat^{n} of $(M, \mathcal{L}|_M)$
  $\int_M c_1(\mathcal{L}) = \beta = n$

- Need $\omega_J|_M$ nondegen. for quantization
  $\Rightarrow$ $n > 0$ (up to orientation)

- Quantization of $M = \mathbb{CP}^1$ w/ $\mathcal{L} = \mathcal{O}(n)$

$\rightsquigarrow \mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$

$= H^0(\mathbb{CP}^1, K^{1/2} \otimes \mathcal{O}(n)) \simeq S^n\mathbb{C}^2$ of dim n

i.e. irred. rep. of $SU(2)$ w/ ht. wt. $j = \frac{n-1}{2}$,

All wt. (= eigenvalue of J_z) : $-j, -j+1, \cdots\cdots, j-1, j$.

- $J^2 = j(j+1) = \frac{n^2-1}{4} = \frac{\mu^2-1}{4}$, as expected.

- Classically, $-\frac{n}{2} \leqslant z \leqslant \frac{n}{2}$ for $z \in M$

vs (qu. mechanical fluctuation).

Quantum, $-\frac{n-1}{2} \leqslant J_z \leqslant \frac{n-1}{2}$ in $\mathcal{H}$

Review § Repr. of non-compact groups (ref. Segal 'Lie groups')

- $G = PSU(1,1) \curvearrowright S^1 \curvearrowright \{\text{funct}^{\text{s}} \text{ on } S^1\}$

$C(S^1)$, $C^\infty(S^1)$, $L^2(S^1)$ should be treated same.

If unitary $\rightsquigarrow$ use Hilbert space (unique).

but $\mathfrak{g} \curvearrowright \times$

- Assume G semi-simple $\curvearrowright V$ topo. v.s.

$G \geqslant K$ max compact. $K \curvearrowright V^{fin}$, $V^{fin} \subset V$ dense

Eg. $PSU(1,1) \curvearrowright C^\infty(S^1), L^2(S^1), C(S^1)$

$PSU(1,1) \geqslant S^1 \curvearrowright \{\text{trigon.}\}$, $\{\text{trigon.}\} \subset C(S^1)$

Fact 1° $\mathfrak{g} \curvearrowright V^{fin}$

2° $G \curvearrowright V$ irred. $\Longrightarrow V^{fin} = \bigoplus_{\rho \in \hat{K}} V_\rho$ as K-mod &, $\dim V_\rho < \infty$

- $G \curvearrowright V \rightsquigarrow (\mathfrak{g}, K) \curvearrowright V^{fin}$

eliminate analysis: algebra remains.

- Eg. Group $G = \mathbb{R} \curvearrowright C^\infty(\mathbb{R})$ translations

$\rightsquigarrow$ $\mathfrak{g}$ acts by $\frac{d}{dx}$, preserving $\overset{\cup\!\mid}{C^\infty(a,b)}$

But $G \not\curvearrowright C^\infty(a,b)$.

- Plancheral theorem.

(i.e. Peter-Weyl theorem if G compact

$L^2(G) \ni f = \sum_P f_P$, $\quad P$: unitary G-mod.

$L^2(G) \ni f = \int f_P \underbrace{d\mu(P)}_{\text{measure on space of irred. repr.}}$

---

§ Representations of $G_{\mathbb{R}} = SL(2,\mathbb{R})$.

$G_{\mathbb{R}}/B = \mathbb{RP}^1 = S^1$

$B \curvearrowright \mathbb{C} \longleftrightarrow \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto (\operatorname{sgn}(a))^\varepsilon |a|^p$

$\varepsilon = 0, 1, \quad p \in \mathbb{C}$

$\rightsquigarrow$ Induced repr. $G_{\mathbb{R}} \curvearrowright E_{p,\varepsilon}$

$\varepsilon = 0$, $E_{p,\varepsilon=0} = \{ f(\theta) |d\theta|^{p/2} \}$ $\quad \frac{p}{2}$-density on S^1

$(\because B \curvearrowright T_e(G/B)$ by $\begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \mapsto a^2)$

$\varepsilon = 1 \longrightarrow$ twisted (by Mobius band).

- $\mathbb{T} = SO(2) \overset{\text{max. cpt.}}{<} G_{\mathbb{R}}$

  $E_{p,\varepsilon}\big|_{SO(2)}$ indep. of p $\quad (\because |a| = 1)$

  $\| $

  $C^\infty(S^1) \ni \varphi \quad \varphi(-z) = (-1)^\varepsilon \varphi(z)$

- $p \in \mathbb{C} \setminus \mathbb{Z} \Rightarrow G_{\mathbb{R}} \curvearrowright E_{p,\varepsilon}$ irred. (not nec. unitary)

- $E_{2,0} = \{\text{density on } S^1\}$

  $\Rightarrow \overline{E_{1+is,\varepsilon}} \otimes E_{1+is,\varepsilon} \xrightarrow{f_1 f_2} E_{2,0} \xrightarrow{\int_{S^1}} \mathbb{C}$ $\quad G_{\mathbb{R}}$-inv. inner product

  $\Rightarrow G_{\mathbb{R}} \curvearrowright E_{1+is,\varepsilon}$ unitary rep.

  Call <u>principal series</u>. (except $E_{1,1}$ reducible)

- <u>Discrete series</u>

  holom. induced from $\mathbb{T} \overset{\text{max. cpt.}}{<} G_{\mathbb{R}}$

  $G_{\mathbb{R}} / \mathbb{T} =: H$ upper half-plane. $G_{\mathbb{R}} = \mathrm{Isom}(H)$

  $\mathbb{T} \curvearrowright \mathbb{C}, \quad z \longmapsto z^p \qquad p \in \mathbb{Z}$

  $\xrightarrow[\text{repr.}]{\text{induced}}$ $G_{\mathbb{R}} \curvearrowright \Omega^{p/2}_{hol.} \ni f(z)(dz)^{p/2}$ $\quad L^2$ holo. $(\frac{p}{2})$-form on H

  $dz \xmapsto{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}} (cz+d)^{-2}\, dz$

- Upper half-plane $H \simeq D$ Unit disk

  $\rightsquigarrow$ $SL(2,\mathbb{R}) \simeq SU(1,1)$

  Taylor series

  $f(z)(dz)^{p/2} = \sum_{n \geq 0} a_n z^n (dz)^{p/2}$

$u \in \mathbb{T} \curvearrowright \mathbb{D}$ rotation $\Rightarrow$ $z^n (dz)^{p/2} \longmapsto u^{n+\frac{p}{2}} \cdot (z^n (dz)^{p/2})$

$\Rightarrow \mathbb{T} \curvearrowright \Omega^{p/2, fin}_{hol.} \simeq \{ \sum_{m \in \frac{p}{2} + \mathbb{N}} a_m e^{im\theta} \}$ trigon poly.

$\rightsquigarrow \mathfrak{g}_{\mathbb{R}} \curvearrowright \Omega^{p/2}_{hol}$ irred. rep. w/ lowest weight $e^{i\frac{p}{2}\theta}$

- Inv. norm $\| f \|^2 = \int_{\mathbb{D}} |f|^2 (1-|z|^2)^{p-2} |dz \, d\bar{z}|$
  $\Rightarrow$ unitary repr.
- $p \leq 1 \Rightarrow \Omega^{p/2}_{hol} \cap L^2 = 0$
- $p = 1 \rightsquigarrow \exists$ another inv. norm $\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$
  $\Rightarrow \Omega^{1/2}_{hol}$ still unitary
- Similar, $\overline{\Omega}^{p/2}_{hol.}$

- $p > 1$, $\Omega^{p/2}_{hol} \oplus \overline{\Omega}^{p/2}_{hol} \subseteq \overbrace{E_{p, \varepsilon(p)}}^{\text{non-unitary}} = \overbrace{\Omega^{p/2}}^{\{\text{all } \frac{p}{2}\text{-forms on } S^1\}}$
- $$V := \frac{\Omega^{p/2}}{\Omega^{p/2}_{hol} \oplus \overline{\Omega}^{p/2}_{hol}} = \bigoplus_{-\frac{p}{2} < m < \frac{p}{2}} \mathbb{C} \langle e^{im\theta} \rangle, \quad \dim V = p-1$$

[Eg. $p=2$, $0 \to \Omega^1_{hol} \oplus \overline{\Omega}^1_{hol} \longrightarrow \Omega^1 \xrightarrow{\int_{S^1}} V = \mathbb{C} \to 0$
In general, $V = (S^{p-2} \mathbb{C}^2)^*$ non-unitary repr. of $SL(2,\mathbb{R})$.]

- Note: Cartan subgp. $A \leq G_{\mathbb{R}}$ (i.e. $A_{\mathbb{C}} \leq G$ as max. torus $\mathbb{C}^\times$)
  $A = \mathbb{R}^\times \rightsquigarrow$ principal series.
  $A = \mathbb{T} \rightsquigarrow$ discrete series.
- Note: $\exists$ complementary series, but of zero measure for Plancherel measure.

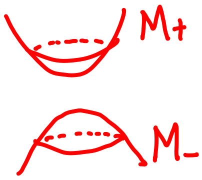
§ Discrete Series of $SL(2,\mathbb{R})$ (double cover of $SO(1,2)$)

$\mathcal{B}'$: $M = Y^{\tilde{\tau}} = \{ z^2 = \frac{\mu^2}{4} + x^2 + y^2 \} \cap \mathbb{R}^3$ (switch $x \mapsto ix$, $y \mapsto iy$)

case (1) $\mu^2 > 0$ ($\Rightarrow \gamma = 0$, $\beta = n \neq 0$)

$$M = M_+ \amalg M_- \curvearrowleft SL(2,\mathbb{R})$$

upper half-plane



$\rightsquigarrow$ $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}'_\pm) \curvearrowleft SL(2,\mathbb{R})$

unitary repr. $D_n^\pm$ w/ $J^2 = (n^2-1)/4$.

- Classically, on M_+, $\frac{n}{2} \leq z < \infty$

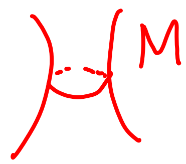
$\rightsquigarrow$ On quantum level D_+, $\frac{n}{2} \lesssim J_z < \infty$.

(Eigenvalues of J_z : $\frac{n+1}{2}$, $\frac{n+1}{2} + 1$, $\frac{n+1}{2} + 2$, ...)

§ Principal Series of $SL(2,\mathbb{R})$

Case (2) $\mu^2 < 0$ ($\Rightarrow \beta = 0$, $\gamma \neq 0$)

$$M = Y^{\tilde{\tau}} = \{ x^2 + y^2 = \frac{\gamma^2}{4} + z^2 \} \sim S^1 \times \mathbb{R}$$



($\rightsquigarrow$ J_z should be unbound in both $\pm\infty$)

- $b_1(M) = 1 \Rightarrow$ 1-parameter family $\mathcal{B}'_\delta$ supp. on M.

- Quantize $M = T^*S^1 \rightsquigarrow$ functions/half densities on S^1.

$$\rightsquigarrow \quad J_z\left(e^{i(n+\delta)\theta}\right) = (n+\delta)\cdot\left(e^{i(n+\delta)\theta}\right)$$

$\mathrm{Spec}(J_z) = \mathbb{Z} + \delta$ (unbound)

- Repr. of $SL(2,\mathbb{R}) \Rightarrow \mathrm{Spec}(J_z) \subset \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2} \Rightarrow \delta = 0$ or $\frac{1}{2}$.

- This is principal series $P_{\gamma,\delta}$.

  $(\gamma,\delta) \neq (0,\frac{1}{2}) \Rightarrow P_{\gamma,\delta}$ irred.

  $(\gamma,\delta) = (0,\frac{1}{2}) \Rightarrow P_{0,\frac{1}{2}} = D_0^+ \oplus D_0^-$

  $\gamma = 0 \sim M = \{z^2 = x^2 + y^2\} = M_+ \cup M_-$ 

  If $\delta \neq \frac{1}{2} \Rightarrow M_+$ & M_- linked by monodromy.

§ Harish-Chandra modules

non-unitary repr

$\sim$ $\mathrm{Spec}(J_z) \subset s + \mathbb{Z}$ w/ $s \in \mathbb{C}$

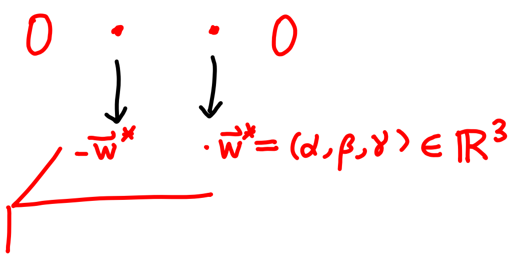
$\sim$ brane, not $\tilde{\tau}$-inv.

Need more branes ......

- Consider branes, $SL(2,\mathbb{R})$-inv. only asym.
- Assume J_z acts diagonally

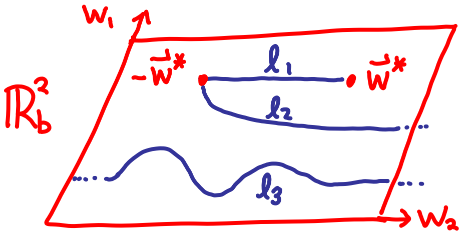
$\sim$ invariant under $U(1) \leq SL(2,\mathbb{R})$

$$\begin{array}{c} U(1) \curvearrowright Y \\ \pi = \vec{w} \Big\downarrow \text{HK moment} \\ \mathbb{R}^3 \end{array}$$



- K-inv. $\iff$ union of fibers of π
- K-inv., τ or $\tilde{\tau}$ & $SL(2,\mathbb{R})$-inv. Lagr. submfd. $= \pi^{-1}(w_2\text{-axis})$
- K-inv. Lagr. submfd. $\iff M = \pi^{-1}(\ell)$, $\ell \subset \mathbb{R}^2 \times b =: \mathbb{R}^2_b \subset \mathbb{R}^3$

Asym. $SL(2,\mathbb{R})$-inv. $\Rightarrow$ $\ell \sim w_2$-axis near ∞.



$M_1 = \pi^{-1}(\ell_1) \sim S^2$

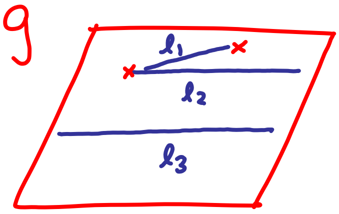
$M_2 \sim \mathbb{R}^2$

$M_3 \sim S^1 \times \mathbb{R}$

$\omega_J|_M$ non-degen $\Rightarrow$ (1) ℓ not a closed curve

(2) $\ell = \{ w_1 = g(w_2) \}$, $\exists$ function g

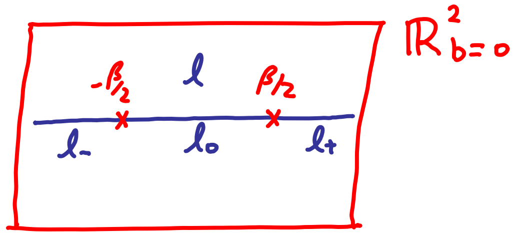
· Up to Hamiltonian diffeo. of Y $\Rightarrow$



§ Complementary Series

corresp. to $\ell = w_2$-axis,

but $\alpha = \gamma = 0$, $\beta > 0$



$\Rightarrow$ $\pm\vec{w}^* = \pm(0, \beta/2, 0) \in \mathbb{R}^3$

$\Rightarrow$ $M = \pi^{-1}(\ell) = \underset{\mathbb{R}^2}{\underset{\wr}{M_-}} \cup \underset{S^2}{\underset{\wr}{M_0}} \cup \underset{\mathbb{R}^2}{\underset{\wr}{M_+}}$

· $\beta = 0$, $\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ principal series as before.

· $\beta \gtrsim 0$, $\mathcal{H}$ irred.; unitary ; $\nexists$ highest / lowest wt. vector.
($\rightsquigarrow$ complementary series of unitary repr. of $SL(2,\mathbb{R})$.

· $\beta = 1$, $\mathcal{H} = \underbrace{\mathcal{H}_-}_{D_1^-} \oplus \underbrace{\mathcal{H}_0}_{\text{1 dim. trivial}} \oplus \underbrace{\mathcal{H}_+}_{D_1^+}$ reducible

non-unique Hermitian structure

· $\beta > 1$, $\mathcal{H}$ irred. (except $\beta \in 2\mathbb{Z} + 1$)

$2n-1 < \beta < 2n+1$: # neg. norm states is n

$\beta = 2n+1$, $\mathcal{H} = \mathcal{H}_- \oplus \underbrace{\mathcal{H}_0}_{\dim 2n+1} \oplus \mathcal{H}_+$

· Remark: $\ell \parallel w_2$-axis in $\mathbb{R}^2_b$

$\Rightarrow$ $M = \pi^{-1}(\ell)$ is (A, B, A)- brane

Can use B-model via Kähler polarizat$^{\underline{n}}$.

· Remark: $\ell \parallel w_1$-axis in $\mathbb{R}^2_b$

$\Rightarrow$ M is (B, A, A)-brane

NOT suitable for quantizat$^{\underline{n}}$.

but A-branes $\sim$ $\mathcal{D}$-modules

(Beilinson-Berstein)

($\sim$ involve solving Hitchin eqt.)

§ $G_{\mathbb{C}}$, other than $SO(3,\mathbb{C})$.

Similar, except $\exists$ nontrivial non-regular coadjoint orbits. But they are related to small representations.

§ CS theory w/ cpt Gauge gp. G (say $\pi_1 = 0$).

C closed Riemann surface

- $\mathcal{M} := \{\text{flat } G\text{-bdl.}/C\}/\cong \;=\mathrm{Hom}(\pi_1(C), G)/\mathrm{Ad}\,G$

  $\exists$ canon. sympl. str. $(\mathcal{M}, \omega_* = F_{\mathcal{L}_*})$

  Aim: Quantize $(\mathcal{M}, k\omega_* =: \omega)$

- $Y := \{\text{flat } G_{\mathbb{C}}\text{-bdl.}/C\}/\cong \;=\mathrm{Hom}(\pi_1(C), G_{\mathbb{C}})/\mathrm{Ad}\,G_{\mathbb{C}}$

  – a natural complexification of $\mathcal{M}$

  "$\{\text{holo. fu. on } Y\}|_{\mathcal{M}} = \{\text{analy. fu. on } \mathcal{M}\}$"

  – $\exists$ holo. sympl. form Ω & $\Omega|_{\mathcal{M}} = \omega$

Aim: A-model on $(Y, \omega_Y = \mathrm{Im}\,\Omega)$

$(\mathcal{M}, \mathcal{L}) \subset Y$ Lagr. A-brane

- $\forall$ cpx. str. $t = J_C$ on C $\xrightarrow[\text{Hitchin}]{}$ complete HK str. on Y
  $t \in \mathcal{T}$ Teichmüller space

$\left(Y \ni \mathbb{C}^r \searrow E \xrightarrow[\text{holo.}]{} C,\ \phi \in H^0(C, K_C \otimes \mathrm{ad}E) \ \text{Higgs stable} \right)$

$\mathcal{H}_t := \mathrm{Hom}(\mathcal{B}_{cc}, (\mathcal{M}, \mathcal{L}))$ a quantizat^{n} of $(\mathcal{M}, \omega)$

$\underset{\text{v.s.}}{=} H^0(\mathcal{M}, \underbrace{\mathcal{L}_*^{\hat{k}} \otimes K^{1/2}}_{\mathcal{L}_*^{k}\ \text{w/}\ \hat{k} = k+h})$ (Fact: $K^{1/2} = \mathcal{L}_*^{-h}$ ← dual Coxeter number)

- flat VB $/ t \in \mathcal{T}$ ($\because$ A-model is inv. under change of HK)

- $\mathrm{Hom}^{\text{cl.}}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) = \mathcal{O}_Y(Y)$ generated by Wilson loop

$\forall$ repr. $G_{\mathbb{C}} \curvearrowright R$ $\forall$ loop $\gamma \subset C$

$W_R(\gamma) = \mathrm{Tr}_R \mathrm{Hol}(\gamma) : Y \longrightarrow \mathbb{C}$ holo. fu.

As $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}) \curvearrowright \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{M}) = \mathcal{H}_t$,

$W_R(\gamma)$ becomes operators on $\mathcal{H}_t$.

§ Other $\mathcal{B}'$

(1) $\mathcal{B}' =$ fiber of Hitchin (cx. Lagr) fibr. $Y \xrightarrow{H} B$

- $H^{-1}(b)$ Abelian var. if b generic

$\Rightarrow \mathcal{H}_b$: quantize $H^{-1}(b) \rightsquigarrow \sim$ Abelian current alg.

- $H^{-1}(0) \supset \mathcal{M}$ as a component (w/ multiplicity)

$\Rightarrow \mathcal{H}_b \supset \mathcal{H}$.

(2) $G_{\mathbb{R}}$ any real form of $G_{\mathbb{C}}$

$G_{\mathbb{R}} = (G_{\mathbb{C}})^{\phi}$ $\exists$ anti-holo. involut$^{\underline{n}}$ ϕ

$\rightsquigarrow \tau_\phi : Y \circlearrowleft \rightsquigarrow \mathcal{M}_\phi := (Y)^{\tau_\phi}$ (up to component)

$\mathcal{M}_\phi = \mathrm{Hom}(\pi_1(C), G_{\mathbb{R}})/Ad(G_{\mathbb{R}})$.

$\rightsquigarrow \widetilde{\mathcal{H}} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{M}_\phi)$ = quantiz. of CS w/ gauge gp. $G_{\mathbb{R}}$.

$\widetilde{\mathcal{H}}_t \simeq H^0(\mathcal{M}_\phi, \mathcal{L}_*^{\hat{k}-h_\phi})$

Some Questions:

Qu: $\mathrm{Hom}(\mathcal{B}_{cc}, M) \overset{?}{\curvearrowleft} \mathrm{Hom}(M, M) = HF_Y(M, M) \underset{Y=T^*M}{\overset{\text{if}}{=\!=}} H_\bullet(\Omega M)$

Qu: $\overset{?}{\nexists}$ analog of bulk or boundary insertion?

Qu: Define $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$, $\mathrm{Hom}(\mathcal{B}_{cc}, \underset{\text{Lagr.}}{\mathcal{B}'})$ by setting up via Costello's approach.

Qu: $\mathrm{Hom}((Y, \omega_J), (Y, \omega_J)) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}})/\mathfrak{f}' \rightsquigarrow$ repr. of $\mathfrak{g}_{\mathbb{C}}$

$(Y, \Omega) \rightsquigarrow$ deform quantizat$^{\underline{n}}$ $\rightsquigarrow U_\hbar(\mathfrak{g}_{\mathbb{C}})$ & repr. of quantum gp.?

Qu: What is SYZ mirror of $\mathcal{B}'$ (in these eg.)

(mirror description of Gukov-Witten, i.e. (B,B,B))

Qu: Tensor product decomp. / branching rules

$\overset{?}{\nexists}$ branes interpretation.

Which $\mathcal{B}'$ corresp. to adjoint repr. ?